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CALCULUS.

237. *Prize Problem. Proposed by S. A. COREY, Hiteman, Iowa.

Find an expression for the remainder after n terms in the following development† of $f(a+x)$:

$$f(a+x)=f(a)+\frac{x}{m.\,2}\Big\{f'(a+x)+f'(a)+2\Big[f'\Big(a+\frac{x}{m}\Big)+f'\Big(a+\frac{2x}{m}\Big)$$

$$+f'\Big(a+\frac{3x}{m}\Big)+\ldots+f'\Big(a+\frac{m-1}{m}x\Big)\Big]\Big\}-\frac{B_1x^2}{m^2.2!}\Big[f''(a+x)-f''(a)\Big]$$

$$+\frac{B_2x^4}{m^4.4!}\Big[f^{\text{iv}}(a+x)-f^{\text{iv}}(a)\Big]-\ldots+(-1)^n\frac{B_nx^{2n}}{m^{2n}.\,(2n)!}\Big[f^{(2n)}(a+x)-f^{(2n)}(a)\Big]$$

$+\ldots$, B_1, B_2, ... being Bernoulli's numbers.

Solution by the PROPOSER.

Taylor's development for $f(a+x)$ may be written,

$$0=f(a)-f(a+x)+xf'(a)+\frac{x^2}{2!}f''(a)+\frac{x^3}{3!}f'''(a)+\ldots+\frac{x^n}{n!}f^{(n)}(a+\theta_1x), \tag{1}$$

where $0<\theta_1<1$.

If $f(b-x)$ be similarly developed, and if b be made equal to $(a+x)$ *after* differentiation, we get,

$$0=f(a+x)-f(a)-xf'(a+x)+\frac{x^2}{2!}f''(a+x)-\frac{x^3}{3!}f'''(a+x)+\ldots$$

$$+(-1)^nf^{(n)}(a+\theta_2x), \tag{2}$$

where $0<\theta_2<1$.

Adding (1) and (2),

$$0=-x[f'(a+x)-f'(a)]+\frac{x^2}{2!}[f''(a+x)+f''(a)]-\ldots$$

$$-\frac{x^n}{n!}[f^{(n)}(a+\theta_2x)-f^{(n)}(a+\theta_1x)], \tag{3}$$

(n odd).

Dividing (3) by x and letting $z_n=[f^{(n)}(a+x)+f^{(n)}(a)]$ and $y_n=[f^{(n)}(a+x)-f^{(n)}(a)]$, we get, by lowering by unity the order of each derivative,

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*In order to emphasize the importance of finding such an expression for the remainder after n terms as will hold good for all integral values of m and approach 0 as m approaches ∞, and at the same time enable computers to determine absolutely in what cases the development holds, the proposer offers a prize of $10 for the best solution. ED. F.

†See *Annals of Mathematics*, Second Series, Vol. 5, No. 4, July, 1904.

$$0=-y_0+\frac{x}{2!}z_1-\frac{x^2}{3!}y_2+\frac{x^3}{4!}z_3-\frac{x^4}{5!}y_5+\dots$$

$$-\frac{x^n}{(n+1)!}[f^{(n)}(a+\theta_2x)-f^{(n)}(a+\theta_1x)], \qquad (4)$$

(n even).

Now develop $f''(a+x)$ by (4), multiply the result by $-(2x^2/4!)$, and add to (4). In the resulting sum, (5), the term involving z_3 has been eliminated.

$$0=-y_0+\frac{x}{2}z_1-\frac{B_1x^2}{2!}y_2+\tfrac{2}{15}\cdot\frac{x^4}{4!}y_4-\tfrac{1}{4}\cdot\frac{x^5}{5!}z_5+\tfrac{5}{14}\cdot\frac{x^6}{6!}y_6-\tfrac{11}{24}\cdot\frac{x^7}{7!}z_7+\tfrac{5}{9}\cdot\frac{x^8}{8!}y_8$$

$$-\tfrac{13}{20}\cdot\frac{x^9}{9!}z_9+\dots+\tfrac{1}{12}\cdot\frac{x^n}{(n-1)!}[f^{(n)}(a+\theta_4x)-f^{(n)}(a+\theta_3x)]$$

$$-\frac{x^n}{(n+1)!}[f^{(n)}(a+\theta_2x)-f^{(n)}(a+\theta_1x)], \qquad (5)$$

B_1 being Bernoulli's first number, $\frac{1}{6}$.

In a similar manner the term in (5) containing z_5 may be eliminated by developing $f^{\text{iv}}(a+x)$ by (4), multiplying result by $+\frac{3x^4}{6!}$ and adding to (5). By continuing this process we may eliminate as many z's as we please, except z_1. Thus to eliminate z_3, z_5, z_7, z_9, and z_{11}, successively, the necessary multipliers are, respectively,

$$-\frac{2x^2}{4!},\ +\frac{3x^4}{6!},\ -\tfrac{20}{3}\cdot\frac{x^6}{8!},\ +\frac{21x^8}{10!},\ \text{and}\ -\frac{90x^{10}}{12!}.$$

The result obtained by eliminating these five z's is found to be identical with the formula given in the problem for the case $m=1$, up to and including the term containing B_5, and may be thus written,

$$0=-y_0+\frac{x}{2!}z_1-\frac{B_1x^2}{2!}y_2+\frac{B_2x^4}{4!}y_4-\frac{B_3x^6}{6!}y_6$$

$$+\frac{B_4x^8}{8!}y_8-\frac{B_5x^{10}}{10!}y_{10}+\frac{x^{12}}{12!}R_{12} \qquad (6)$$

$$\text{where } R_{12}=-12!\left(\frac{\varepsilon_1}{13!}-\frac{2\varepsilon_2}{4!\,11!}+\frac{3\varepsilon_3}{6!\,9!}-\frac{20\varepsilon_4}{3.8!\ 7!}+\frac{21\varepsilon_5}{10!\,5!}-\frac{90\varepsilon_6}{12!\,3!}\right), \qquad (7)$$

the ε's having some value between the greatest and least value of $[f^{12}(a+\theta_1x)-f^{12}(a+\theta_2x)]$.

Let r_{12} be positive and equal to the greatest numerical value which ϵ may take. Then as

$$12!\left(\frac{1}{13!}+\frac{2}{4!\,11!}+\frac{3}{6!\,9!}+\frac{20}{3.8!\,7!}+\frac{21}{10!\,5!}+\frac{90}{12!\,3!}\right)r_{12}=\tfrac{27478}{455}r_{12},$$

$$-\tfrac{27478}{455}r_{12}<R_{12}<+\tfrac{27478}{455}r_{12},.$$

where the coefficient of r_{12} is obtained by changing all the negative signs in (7) to positive, substituting unity for each of the ϵ's and adding.

If now

$$f\left(a+\frac{x}{m}\right), f\left[\left(a+\frac{x}{m}\right)+\frac{x}{m}\right], f\left[\left(a+\frac{2x}{m}\right)+\frac{x}{m}\right], \ldots, f\left[\left(a+\frac{m-1}{m}x\right)+\frac{x}{m}\right],$$

be developed successively by (6), simplifying each by the aid of the preceding, the result is identical with the formula given in the problem up to and including B_5, and may be thus written,

$$f(a+x)=f(a)+\frac{x}{m.2}\left[f'(a+x)+f'(a)+2\left(f'(a+\frac{x}{m})+f'(a+\frac{2x}{m})+\ldots\right.\right.$$

$$\left.\left.+f'(a+\frac{m-1}{m}x)\right)\right]-\frac{B_1x^2}{m^2.2!}[f''(a+x)-f''(a)]+\frac{B_2x^4}{m^4.4!}[f^{\text{iv}}(a+x)-f^{\text{iv}}(a)]$$

$$-\frac{B_3x^6}{m^6.6!}[f^{\text{vi}}(a+x)-f^{\text{vi}}(a)]+\frac{B_4x^8}{m^8.8!}[f^{\text{viii}}(a+x)-f^{\text{viii}}(a)]$$

$$-\frac{B_5x^{10}}{m^{10}.10!}[f^{\text{x}}(a+x)-f^{\text{x}}(a)]\pm\frac{x^{12}.m.R_{12}}{m^{12}.12!}. \qquad (8)$$

By a close inspection of the method used to eliminate the z_3, z_5, z_7, z_9, and z_{11} from (4), we learn that the following formula may be generally used to obtain the multiplier, $\frac{c_sx^{(s-1)}}{(s+1)!}$, which may be employed to eliminate z_s after all the preceding z's except z_1 have been eliminated, viz.,

$$0=S!\left[\frac{1}{(s+1)!}-\frac{2}{4!\,(s-1)!}+\frac{3}{6!\,(s-3)!}-\frac{20}{3}\cdot\frac{1}{8!\,(s-5)!}\right.$$

$$\left.+\frac{21}{10!\,(s-7)!}-\frac{90}{12!\,(s-9)!}+\ldots+\frac{c_s}{(s+1)!2}\right]. \qquad (9)$$

which must hold for *all odd* values of s except $s=1$. It can also be shown that when s is even, say $2n$, (9) gives the value of $\pm B_n$ according as n is odd or even.

After thus eliminating z_3, z_5, z_7, z_9, ... and z_s from (4), and repeating the process used to obtain (8), we obtain a development which is identical with the development up to and including the term containing B_n where $n=\frac{s-1}{2}$. That this is true for *all* values of n follows from the fact that the two developments become identical when $R=0$ for *any* value of n.

Likewise by a method similar to that used to obtain R_{12}, we may obtain the value of $R_{(2n+2)}$. That method is: Find the sum of all the terms in (9) after making all negative signs positive, and after making $s=(2n+2)$, observing that *no new term is to be added.* Calling this sum $t_{(2n+2)}$ we may write

$$R_{(2n+2)}=\pm t_{(2n+2)}\,[f^{(2n+2)}(a+\theta_1 x)-f^{(2n+2)}(a+\theta_2 x)],$$

where both θ_1 and θ_2 have some value greater than zero and less than unity, but not the same value unless $R=0$.

The remainder after the term involving B_n may therefore be expressed as follows:

$$\pm\frac{x^{(2n+2)}.t_{(2n+2)}}{m^{(2n+1)}.(2n+2)!}[f^{(2n+2)}(a+\theta_1 x)-f^{(2n+2)}(a+\theta_2 x],^*$$

where the value of $t_{(2n+2)}$ must be computed for higher values of n than are here given, viz.,

$$t_4=\tfrac{8}{15},\ t_6=\tfrac{8}{7},\ t_8=\tfrac{148}{45},\ t_{10}=\tfrac{410}{33},\ t_{12}=\tfrac{27478}{455},\ t_{14}=\tfrac{3340}{9},\text{ and } t_{16}=\tfrac{241048}{85}.$$

To obtain such higher values of t it is, of course, necessary first to find values of c_s in (9) higher than the following, which were used to find the foregoing values of t, viz.,

$$c_3=-2,\ c_5=+3,\ c_7=-\tfrac{20}{3},\ c_9=+21,\ c_{11}=-90,\ c_{13}=+\tfrac{7601}{15},\text{ and } c_{15}=-3640.$$

The foregoing proof is based on the assumption that all the derivatives are finite and continuous between the limits $x=0$ and $x=x$.

It will be observed that the foregoing solution does not give the value of the remainder after the *general* term as required in the problem, the value of t not being given for more than the first few terms. In actual work the values of t here given are probably all that the computer needs. But, if not, higher values may be found without excessive labor in the manner given. This solution, therefore, places the formula given in the problem on a fairly satisfactory basis in so far as the needs of the computer are involved.

As the Method of Approximation which the writer gave in the June-July, 1906, number of the MONTHLY is based on the formula of the problem,

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*The quantitity within the brackets cannot exceed twice the *greatest* variation of $f^{(2n+2)}(a+\theta x)$ for *any single interval* (x/m) as x increases from $x=0$ to $x=x$.

this solution will serve to determine the limits within which errors in results due to the dropping of higher terms must lie whenever it is possible to determine the limits within which the value of the lowest dropped derivative lies.

NOTE. Mr. Corey informs us that he did not succeed in obtaining a solution until a month after the problem was published, though he had repeatedly attempted to do so. Mr. Corey's solution is quite satisfactory so far as its use for practical computation is concerned, yet greater value would result were an expression found for the remainder after the general term. A prize of $10 is still offered by Mr. Corey for finding an expression for the remainder after the nth term. ED. F.

238. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Solve the differential equations

$$(a)\ (x+y^2)dx+(x^2+y)dy=0,$$
$$(b)\ (x+xy+y^2)dx-(x^2+xy-y)dy=0.$$

Solution by GEORGE W. HARTWELL, Columbia University, New York.

(a) $(x+y^2)dx+(x^2+y)dy=0$.

This equation can be written

$$\frac{dx}{x^2+y}=\frac{dy}{-x-y^2};\ \therefore\frac{dx+dy}{(x-y)\ (x+y-1)}=\frac{xdy+ydx+dx+dy}{(x-y)(xy-1)}.$$

Multiplying both members by $x-y$,

$$\frac{dx+dy}{x+y-1}=\frac{xdy+ydx+dx+dy}{xy-1}.$$

Clearing of fractions and transposing,

$$(x+y-1)(xdy+ydx)-(xy-1)(dx+dy)+(x+y-1)(dx+dy)=0.$$

Dividing by $(x+y-1)^2$,

$$\frac{(x+y-1)\ (xdy+ydx)-(xy-1)\ (dx+dy)}{(x+y-1)^2}+\frac{dx+dy}{x+y-1}=0.$$

Integrating, $\dfrac{xy-1}{x+y-1}+\log(x+y-1)=a$. Multiplying by $(x+y-1)$,

$$xy-1+(x+y-1)\log(x+y-1)=a(x+y-1).$$

(b) $(x+xy+y^2)dx-(x^2+xy-y)dy=0$.

This equation can be written

$$\frac{dx}{x^2+xy-y}=\frac{dy}{y^2+xy+x};\ \therefore\frac{dx-dy}{(x+y)\ (x-y-1)}=\frac{xdx+ydy}{(x+y)\ (x^2+y^2)}.$$

Multiplying by $(x+y)$, $\dfrac{dx-dy}{x-y-1}=\dfrac{xdx+ydy}{x^2+y^2}$. Integrating,

$$2\log(x-y-1)=\log(x^2+y^2)+\log a;\ \therefore (x-y-1)^2=a(x^2+y^2).$$

Also solved by G. B. M. Zerr.